INTERSECTION OF THREE PLANES

Intersection of Three Planes

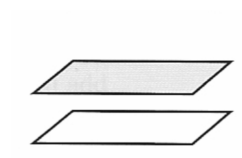
When investigating the intersection of three planes, many possibilities may occur.

Type #1 – The Normal Vectors of All Three Planes are Parallel

Example
Three Planes are Identical

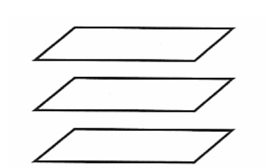
π_1: $x + y + z + 2 = 0$
π_2: $2x + 2y + 2z + 4 = 0$
π_3: $3x + 3y + 3z + 6 = 0$

Example
Two Planes are Identical and Other Plane is Parallel



π_1: $x + 3y + 5z - 10 = 0$
π_2: $2x + 6y + 10z - 18 = 0$
π_3: $x + 3y + 5z - 9 = 0$

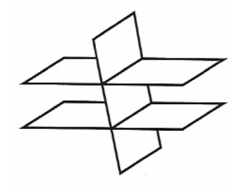
Example
Three Planes are Parallel and None are Identical



π_1: $x + y + z + 2 = 0$
π_2: $2x + 2y + 2z + 5 = 0$
π_3: $x + y + z - 3 = 0$

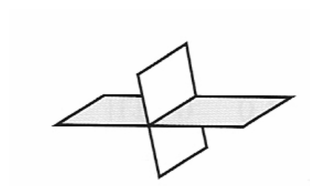
Type #2 – Only Two of the Normal Vectors are Parallel

Example
Two Planes are Parallel and Distinct; the Other Plane is Not Parallel



π_1: $3x+5y-2z-4=0$
π_2: $4x-7y+6z+11=0$
π_3: $6x+10y-4z+1=0$

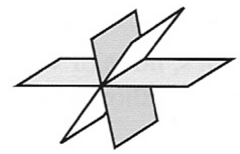
Example
Two Planes are Identical; the Other Plane is Not Parallel



π_1: $3x+5y-2z-4=0$
π_2: $6x+10y-4z-8=0$
π_3: $4x-7y+6z+11=0$

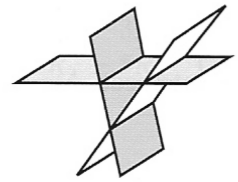
Type #3 – None of the Normal Vectors are Parallel

Example
Three Planes Intersect in a Single Line



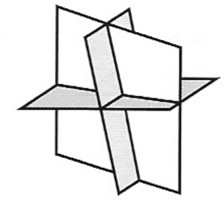
π_1: $x + 2y - 3z - 4 = 0$
π_2: $5x + y + 2z - 7 = 0$
π_3: $7x + 5y - 4z - 15 = 0$

Example
Three Planes Intersect in Pairs, Forming Three Parallel Lines



π_1: $x + 2y - 3z - 4 = 0$
π_2: $5x + y + 2z - 7 = 0$
π_3: $7x + 5y - 4z - 18 = 0$

Example
Three Planes Intersect at a Single Point



π_1: $3x+2y-z-13=0$
π_2: $x-y+2z+3=0$
π_3: $2x+3y-4z-17=0$

A Quick Way to Check if Three Planes Intersect at a Single Point

Notice that when three planes intersect at a single point (as shown on the previous page), their normal vectors are not coplanar. In fact, this case is the **only** one in which the three normal vectors are not coplanar.

- Therefore, a quick way to check if three planes intersect at a single point is to see whether or not their normal vectors are coplanar.
  - Recall that if $\vec{u} \bullet \vec{v} \times \vec{w} = 0$, then $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar.
    - If $\vec{u} \bullet \vec{v} \times \vec{w} \neq 0$, then $\vec{u}$, $\vec{v}$ and $\vec{w}$ are **not** coplanar.

Checking if Three Planes Intersect at a Single Point

Suppose three planes have normal vectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$.

- If $\vec{n}_1 \bullet \vec{n}_2 \times \vec{n}_3 \neq 0$, then the normal vectors are **not** coplanar and the three planes intersect at a single point.
- If $\vec{n}_1 \bullet \vec{n}_2 \times \vec{n}_3 = 0$, then the normal vectors are coplanar and the three planes do **not** intersect at a single point (they may intersect in another way though).

Example
Determine if the following planes intersect at a single point.

π_1: $2x - y + 3z - 2 = 0$
π_2: $x - 3y + 2z + 10 = 0$
π_3: $5x - 5y + 8z + 3 = 0$